1367

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

EDITED BY

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OF THE REGISTRAR
3 DEC 1924

Volume XV, No. 10.

August 1924.

562

MADRAS:

PRINTED BY SRINIVASA VARADACHARI & CO.

Annual Subscription, Rs. 6.] [Single Copy, One Rupee.

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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Contributors to Part I will be presented 25 copies of reprints of their contributions. Extra copies will be supplied, if desired, at net cost.

All contributions should be written legibly on one side only of the paper and all diagrams should be neatly and accurately drawn on separate slips.

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2. Next, let us consider how a straight line has *two directions.*

In the above figure, let OA $= r$. Then OA$' = -r$. If t be a particle of time and r be the corresponding particle of space, then

$$t = kr^2,$$

where k is a constant. (See *Journal of the Indian Mathematical Society*, Vol. XV, No. 1, page 10).

$$\begin{aligned} \text{Hence} \quad t \text{ of OA} &= kr^2 \\ &= k(-r)^2 \\ &= t \text{ of OA}'. \end{aligned}$$

That is, the time value corresponding to two equal lengths along OX, OX$'$ is the same.

Hence if we draw the lines of time corresponding to the straight lines OX, OX$'$, we get the same line of time.

Hence the straight lines which can be drawn from O in a Euclidean space continuum of a three dimensional world are arranged in pairs each yielding one line in the equivalent time continuum. Such straight lines we define as parts of the same straight line of different directions.

So also in a Euclidean three dimensional continuum of matter in a four-dimensional continuum, straight lines through a point O are arranged in trios, each yielding one line in the equivalent time continuum.

For, $\quad \mathrm{M}^3 = (\wedge \mathrm{M})^3 = (\text{A}\mathrm{M})^3$

and each gives the same t. ($\wedge$, A are dinus and trinus signs. See *J.I.M.S.*, Vol. XV, p. 28).

Thus in Time, Space and Matter continua, Euclidean under the guidance of a perpendicular direction, there are straight lines radiating from a given point which are all of different directions in time, which are arranged in pairs of same direction in space and which are arranged in trios of the same direction in matter.

(i) A straight line in time has one direction : positive.

(ii) A straight line in space has two directions: positive and negative.

(iii) A straight line in matter has three directions: positive, dinitive and trinitive.

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A Classical Indian Puzzle-Problem.

By A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.,

1. In the *Messenger of Mathematics* Vol. LIII., No. 2, Dr. Glaisher has discussed some of the puzzle-problems of Widmont and Blasius. This reminds us of an interesting variant of those problems occurring in some of the classical Indian treatises on Arithmetic, such as *Bijaganita* of Bhâskara and *Ganitasara-sangraha* of Mahâvirâchârya. Both these works are about a thousand years old; but the problem we shall refer to is perhaps of a much more ancient origin; for, we find Bhaskara (about 1050 A. D.) citing it as one instanced by earlier authors.

The problem is this:

Three traders, with six, eight and hundred *panas* as capital, respectively, having bought fruits at a uniform rate, sold a part in lots and disposed of the remainder at one for five *panas* and thus became *equally* rich. What were the purchase and sale prices ?

The rule for solving such problems is enunciated by Vishnugupta, Mahaviracharya and others, thus :

The required sale-price is any number greater than the greatest of the given capitals, while the purchase-price is less than the product of the two sale-prices by unity.

Bhaskara, however, attempts the solution of a set of linear indeterminate equations using the Indian *kuttaka* or continued fraction method and finally arrives at the correct result, though in the course of his reasoning he makes serious mistakes.

The solution (given by Vishnugupta and others) is the same as that given by Dr. Glaisher's '*quotient*' system (so called, because the figures in the second column, *vide infra*, are the quotients obtained by dividing those in the first column by p) on p. 19, Vol. LIII, No. 2, *Messenger of Mathematics* :

$p\mu-1$	$\mu-1,\ p:\ p-1;\ \mu$	$p\mu-1$
$2(p\mu-1)$	$2\mu-1,\ p:\ p-2;\ \mu$	,,
....	....	....
$p(p\mu-1)$	$p\mu-1,\ p:\ 0:\ \mu$	,,

In the above tables μ corresponds to the Indian residual (retail) sale-price, p articles per *pana* is the sale-price in lots, and $(p\mu - 1)$ articles per *pana* the purchase-rate, while the co-efficients of $(p\mu - 1)$ in the first column represent the capitals and the numbers in the last column the common realised amount.

The General Problem.

2. We shall now discuss the general problem of n capitals ($c_1, c_2, \ldots\ldots c_n$, in ascending order of magnitude) and the residual sale-price μ and give all the classes of solutions in terms of the '*quotient*', as well as the '*non-quotient*' systems.

Denote the purchase-price by x, the sale-price in lots by p, the number of articles sold in lots by the several tradesmen by $pa_1, pa_2, \ldots\ldots pa_n$, and the corresponding residual articles by $b_1, b_2, \ldots\ldots b_n$; and the problem reduces itself to the solution, in positive integers, of the following $(2n - 1)$ indeterminate equations :—

$$\left.\begin{aligned} c_1x = pa_1 + b_1;\ c_2x = pa_2 + b_2;\ \ldots\ldots c_nx = pa_n + b_n; \\ a_1 + \mu b_1 = a_2 + \mu b_2 = \ldots\ldots = a_n + \mu b_n \end{aligned}\right\} \quad \text{(A)}$$

where $$c_1 < c_2 < c_3 \ldots\ldots < c_n.$$

From the equations (A), we derive

$$\left.\begin{aligned} (b_1 - b_2)(p\mu - 1) &= (c_2 - c_1)\,x \\ (b_2 - b_3)(p\mu - 1) &= (c_3 - c_2)\,x \\ &\ldots\ldots\ldots\ldots \\ (b_{n-1} - b_n)(p\mu - 1) &= (c_n - c_{n-1})\,x \end{aligned}\right\} \quad \ldots \quad \ldots \text{(B)}$$

By addition,

$$(b_1 - b_n)(p\mu - 1) = (c_n - c_1)\,x$$

Let h be the H. C. F. of $c_2 - c_1, c_3 - c_2, \ldots\ldots c_n - c_{n-1}$

Then hx will be divisible by $p\mu - 1$.

Set $$hx = k(p\mu - 1).$$

Then $$b_1 - b_n = \frac{(c_n - c_1)\,k}{h}.$$

$$\therefore \quad b_1 \geqslant \frac{(c_n - c_1)\,k}{h}. \quad \ldots \quad \ldots \text{(1)}$$

Since $$c_1 x = pa_1 + b_1,$$

$$c_1 x \geqslant b_1;$$

i. e., $$\frac{c_1 k(p\mu - 1)}{h} \geqslant \frac{(c_n - c_1)\,k}{h}.$$

$$\therefore \quad p \geqslant \frac{c_n}{c_1 \mu}. \quad \ldots \quad \ldots \text{(2)}$$

Now, by giving values 1, 2, 3, ... for k and for p any integral value not less than $\frac{c_n}{c_1\mu}$, so that

$$k(p\mu-1) \equiv 0 \pmod h,$$

the corresponding values of x may be determined and hence those of a_1 and b_1 from the equations

$$pa_1 + b_1 = c_1 x = \frac{k(p\mu - 1)c_1}{h}.$$

Obviously, a_1 can take all values 0, 1, 2, ... up to

$$I\left\{\frac{1}{p}\left[\frac{k(p\mu-1)c_1}{h} - \frac{(c_n - c_1)k}{h}\right]\right\}$$

i. e.,

$$I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\},$$

where $I(m)$ represents the integral part of m.

The values of $(a_1 + \mu b_1)$ vary with those of a_1 and b_1.

Thus, when

$$a_1 = r,$$

$$b_1 = \frac{k(p\mu - 1)c_1}{h} - pr,$$

and

$$a_1 + \mu b_1 = (p\mu - 1)\left(\frac{\mu k c_1}{h} - r\right).$$

The least value of $(a_1 + \mu b_1)$ occurs when a_1 is a maximum,

i. e., when

$$a_1 = I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\}.$$

When a_1, b_1, are determined, equations (B) give the values of b_2, b_3, ... from which the corresponding values of a_2, a_3, ... are readily gathered.

Thus the equations (A) are completely solved.

Note: If d be the highest common factor of μ and h, k also must be a multiple of d and so k cannot take all values 1, 2, 3, ...

3. *Classification of Solutions*:

(i) If

$$\frac{(c_n - c_1)k}{h} \geqslant p \geqslant \frac{c_n}{c_1\mu},$$

the corresponding solutions belong entirely to the non-quotient system, since $b_1 \geqslant p$; and for a given set of values k and p (satisfying, of course, the limitations of § 2) there are

$$1 + I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\}$$

non-quotient systems satisfying the equations (A).

(ii) If p be greater than both $\frac{c_n}{c_1\mu}$ and $\frac{(c_n - c_1)k}{h}$, there may be, for a given set of values of k and p, one quotient system and

$$I\left\{\frac{k(p\mu - 1)c_1}{ph}\right\}$$

non-quotient systems, or merely

$$I\left\{\frac{k(p\mu - 1)c_1}{ph}\right\}$$

non-quotient systems satisfying (A).

(iii) If $k\mu c_1 \leqslant h$, it is easily seen that

$$I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\}$$

vanishes and therefore there is *only one* solution in the quotient system and *none* in the non-quotient system.

4. *Illustration*:

In the problem cited by Bhaskara, $c_1 = 6$, $c_2 = 8$, $c_3 = 100$, $\mu = 5$.

The H. C. F. of $(c_2 - c_1)$ and $(c_3 - c_1)$ is 2, *i.e.*, $h = 2$;

and $$p \geqslant \frac{100}{6 \times 5} \text{ or } \geqslant 4;$$

and $$b_1 \geqslant \frac{(100 - 6)k}{2} \text{ or } 47k.$$

(i) When $47k \geqslant p \geqslant 4$, the solution is exclusively in the non-quotient systems; thus let $k = 1$, then since $x = \frac{5p - 1}{2}$ = an integer, $v \equiv 1 \pmod 2$ and $p = 5$ is the least value admissible; there are

$$1 + I\left\{\frac{1}{5 \times 2}(5 \times 5 \times 6 - 100)\right\}$$

or 6 non-quotient systems which are solutions of the problem (*vide* § 3 (i)). These are exhibited below in a tabular form:—

[C = capital; P = purchase-price; F = number of fruits bought; L = number sold in lots; R = residual articles; A = amount realised.]

Non-Quotient Systems.

Serial No.	C	P = x	F = CP	L	R	$A = 5R + \frac{L}{p}$
1	6	12	72	0	72	360
	8	12	96	25	71	360
	100	12	1200	1175	25	360
2	6	12	72	5	67	336
	8	12	96	30	66	336
	100	12	1200	1180	20	336
3	6	12	72	10	62	312
	8	12	96	35	61	312
	100	12	1200	1185	15	312
4	6	12	72	15	57	288
	8	12	26	40	56	288
	100	12	1200	1190	10	288
5	6	12	72	20	52	264
	8	12	96	45	51	264
	100	12	1200	1195	5	264
6	6	12	72	25	47	240
	8	12	96	50	46	240
	100	12	1200	1200	0	240

(ii) When $p \geqslant 4$ and also $> 47\,k$, there may or may not be a solution in the quotient system. For, let $k = 1$ and $p = 49$; then

$$x = \frac{49 \times 5 - 1}{2} = 122.$$

Now proceeding to solve the equations

$$49\,a_1 + b_1 = 6x = 732,$$

we find that for a solution in the quotient system, b_1 is equal to the least positive remainder obtained by dividing 732 by 49.

$$\therefore \qquad b_1 = 46,$$

which does not satisfy the condition

$$b_1 \geqslant 47\,k,$$

since $\quad k = 1.$

Thus there is no solution in the quotient systems for $k = 1, p = 49$.

In this case also, therefore, the solutions are all in the non-quotient systems of which there are

$$I\left(\frac{49\times 5-1}{2\times 49}\times 6\right)$$

or 14 according to the formula of § 3. (ii).

These correspond to the values 0, 1, 2, 3, ...13 for a_1 and the realised amounts

3660, 3416, 3172, 2928, 2684, 2440,
2196, 1952, 1708, 1464, 1220, 976, 732, 488 and 244.

If however $k = 1$, $p = 51$, then $x = 127$ and the remainder obtained by dividing 762 by 51 is 48, which is greater than 47, and hence there is one solution in the quotient-system, *viz.*

C	P = x	F = CP	L	R	$A = 5R + \frac{L}{p}$.
6	127	762	714	48	254
8	127	1016	969	47	254
100	127	12700	12699	1	254

and $I\left(\frac{762}{51}\right)$ *i.e.*, 14 solutions in the non-quotient system, corresponding to the realised amounts

508, 762, 1016, 1270, 1524, 1778, 2032,
2286, 2540, 2794, 3048, 3302, 3556 and 3810.

5. For the existence of a solution in the quotient-system, it is necessary and sufficient, that the remainder obtained by dividing

$$\frac{k(p\mu-1)c_1}{h} \text{ by } p$$

should be greater than or equal to

$$\frac{(c_n-c_1)k}{h}.$$

It is not possible to obtain an expression for the form of this remainder except in the following two cases :—

Case I. If $\frac{kc_1}{h} < p$ and is an integer, the remainder is easily seen to

be $p - \frac{kc_1}{h}$ and hence the necessary and sufficient condition for a solution in the quotient-system is in this case

$$p - \frac{kc_1}{h} \geqslant \frac{(c_n - c_1)k}{h};$$

i.e.

$$p \geqslant \frac{c_n k}{h},$$

in addition to the condition that $\frac{kc_1}{h}$ is an integer.*

These conditions are evidently satisfied by taking

$$k = h \text{ and } p \geqslant c_n.$$

Then

$$x = p\mu - 1.$$

This is just the Indian solution, where $p > c_n$ in order to avoid the zero-remainder. Thus, in Bhaskara's problem, if we take

$$k = 2, p = 100, \text{ then } x = 499;$$

and the solutions represented below involve a zero-remainder :—

C	P	F	L	R	$A = 5R + \frac{L}{p}$.
6	499	2994	2900	94	499
8	499	3992	3900	92	499
100	499	49900	49900	0	499

while if $k = 2, p = 101$, then $x = 504$

and the solutions represented below have no such zero-remainder.

C	P	F	L	R	$A = 5R + \frac{L}{p}$.
6	504	3024	2929	95	504
8	504	4032	3939	93	504
100	504	50400	50399	1	504

This is, in fact, the solution given by Bhaskara.

* The two conditions $p \geqslant \frac{c_n k}{h}$ and $\frac{kc_1}{h}$ = an integer imply the other condition $p \geqslant \frac{c_n}{c_1 \mu}$, for $kc_1 > h$ and therefore $kc_1\mu > h$, that is, $c_n\frac{k}{h} > \frac{c_n}{c_1\mu}$.

Case II. If $\frac{kc_1}{h} < p$ but not an integer, let h_1 be the remainder when $k\mu c_1$ is divided by h; then the remainder when $\frac{k(p\mu - 1)c_1}{h}$ an integer is divided by p is either

$$\frac{ph_1 - kc_1}{h}, \text{ when } ph_1 > kc_1;$$

or

$$p + \frac{ph_1 - kc_1}{h}, \text{ when } ph_1 < kc_1.$$

The following sets of conditions are necessary and sufficient for a quotient system in this case :—

(1) $$p \geqslant \frac{kc_n}{h_1};\ p \geqslant \frac{c_n}{c_1\mu}$$

which is included in the former when $k\mu c_1 \geqslant h$ (*vide* § 2).

or (2) $$p \geqslant \frac{kc_n}{h+h_1};\ p < \frac{kc_1}{h_1};\ p \geqslant \frac{c_n}{c_1\mu};\ p > \frac{kc_1}{h}.$$

The latter conditions imply the further relation

$$\frac{kc_n}{h+h_1} < \frac{kc_1}{h},\ i.e.,\ c_n < c_1\left(1 + \frac{h}{h_1}\right).$$

Cor.: (i) p cannot take any value between

$$\frac{kc_1}{h_1} \text{ and } \frac{kc_n}{h_1}.$$

(ii) If $\frac{c}{c_1} < 1 + \frac{h_1}{h}$ and $k\mu c_1 > h$, it follows easily that $\frac{kc_1}{h}$ is greater than both $\frac{kc_n}{h+h_1}$ and $\frac{c_n}{c_1\mu}$. Thus, in this case the total number of values of p between $\frac{kc_1}{h}$ and $\frac{kc_1}{h_1}$ giving the quotient systems are

$$I\left\{\frac{kc_1(h - h_1)}{h^2h_1}\right\}, \text{ or } 1 + I\left\{\frac{kc_1(h - h_1)}{h^2h_1}\right\}.$$

Illustration.

Let $c_1 = 100,\ c_2 = 113,\ c_3 = 139,\ \mu = 20.$

Then $h = 13$; giving k the value 10, we notice that p should satisfy the equation

$$20\,p \equiv 1 \text{ (mod. 13)},$$

i.e. $$p \equiv 2 \text{ (mod. 13)}.$$

Now $\frac{kc_1}{h} = \frac{1000}{13} = 76\frac{12}{13}$; $h_1 = 6$; $\frac{kc_1}{h_1} = 166\frac{2}{3}$;

$$\frac{kc_n}{h+h_1} = 73\frac{3}{19}; \quad \frac{c_n}{c_1\mu} = \frac{139}{2000}; \quad \frac{kc_n}{h_1} = 231\frac{2}{3}.$$

The values of p greater than $\frac{kc_1}{h_1}$ admissible in this problem are the integers greater than 231 which are congruent to 2 (mod. 13), *viz.*, the infinite series in A.P with common difference 13, beginning with 236.

For example, when $p = 236$, $x = 3630$ and the solution is as follows:

C	P	F	L	R	A
100	3630	363000	362968	32	2178
113	"	410190	410168	22	"
139	"	504570	504568	2	"

The values of p greater than $\frac{kc_1}{h}$ and less than $\frac{kc_1}{h_1}$ are given by the integers between $76\frac{12}{13}$ and $166\frac{2}{3}$ which are congruent to 2 (mod. 13) *viz.*, the seven integers

80, 93, 106, 119, 132, 145, 158.

NOTE 1. It is obvious that when $\frac{kc_1}{h} = p$ or a multiple of p, the least value of $b_1 = 0$ and there is no solution in the quotient system.

NOTE 2. When $\frac{kc_1}{h} > p$, it is not possible to define the limits of p in terms of the constants $k, h, h_1, \mu, c_1, c_2, \ldots\ldots$ and therefore the existence of a solution in the quotient system has to be verified only by trial. Thus for the problem discussed above, the values of

$$p < \frac{kc_1}{h} \text{ or } 76\frac{12}{13}$$

which are likely to give a solution in the quotient-system are the integers congruent to 2 (mod. 13) and lying between

$$\frac{(c_3 - c_1)k}{h} \text{ and } \frac{kc_1}{h}$$

i.e., between 30 and $76\frac{12}{13}$, *viz.,* 41, 54 and 67.

But it can be verified that the remainder obtained by dividing $\frac{k(p\mu - 1)c_1}{h}$, *i.e.,* $\frac{1000(20p - 1)}{13}$ by p is not greater than 30 for any of the above values of p.

Thus the only solutions in the quotient-system corresponding to the value 10 of k are those in which $p = 80, 93,$ ...... up to 158 and the terms in the infinite A.P with common difference 13, beginning with 236; 80 is therefore the least value of p when $k = 10$.

Summable Double Series.*

By V. TIRUVENKATACHARYA, M.A., L.T.

1. Professors Bromwich and Hardy have shown in the *Proceedings* of the London Mathematical Society, Series 2, Vol. 2, that, by extending the theorems of Frobenius and Holder to double series, it is possible to find the sum of the oscillatory series

$$\begin{array}{l} 1-1+1-1+1-\ldots\ldots \\ -1+1-1+1-1+\ldots\ldots \\ +1-1+1-1+1-\ldots\ldots \\ -1+1-1+1-1+\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

and have obtained the sum of this series to be $\frac{1}{4}$. In this paper, it is attempted to develop a general theory by which we can find the sum of this and other oscillatory double series. The discussion in the following pages closely follows the treatment in §§ 99—103 of Bromwich in his *Theory of Infinite Series.*

2. GENERAL THEORY :—*Double Integral for Summable Double Series.* [The double integral being of the same type as Borel's integral for single summable series, the integral may be called Borel's double integral.]

Following Prof. Hardy in his definition of single summable series, let us enunciate the following principle† :— If two limiting processes, performed in a definite order on a function lead to a definite value A, but, when performed in a reverse order lead to a meaningless result B, then we may agree to interpret B as meaning A.

For example, the equation

$$\lim_{(m,n)\to\infty}\left[\int_0^\infty\int_0^\infty f(x,y,m,n)\,dx dy\right] = \int_0^\infty\int_0^\infty\left[\begin{matrix} f(x,y,m,n) \\ (m,n)\to\infty\end{matrix}\right] dx dy \quad 2.1$$

cannot be true always. But it may happen that the expression on the right is perfectly definite, while that on the left is not. We may agree, then, according to the principle enunciated above, to take the right-hand integral as defining the left-hand limit.

---

* Read before the Fourth Conference of the Indian Mathematical Society, Poona, April 1924.

† *Vide* Bromwich : *Theory of Infinite Series*, p. 267.

[Here $f(x, y, m, n)$ is a function of four variables and the two limiting processes, are (*a*) making m and n tend to ∞ and (*b*) the double integration with respect to x and y. The conditions which $f(x,y,m,n)$ must satisfy in order that the double integral may be unique, irrespective of the order of integration, and the double limit unique, irrespective of the order of making m and n tend to infinity, will be specified in the course of the paper.]

Suppose that $f(x, y, m, n)$ is the sum of $(m + 1)$ rows and $(n + 1)$ columns of a double series of terms which are functions of x and y.

$$\begin{aligned} \text{Let } f(x, y, m, n) = {} & \phi_{00}(x, y) + \phi_{01}(x, y) + \quad \ldots \quad + \phi_{0n}(x, y) \\ & + \phi_{10}(x, y) + \quad \ldots \quad \ldots \quad + \phi_{1n}(x, y) \\ & + \quad \ldots \quad \ldots \quad \ldots \\ & + \phi_{m0}(x, y) + \quad \ldots \quad \ldots \quad + \phi_{mn}(x, y) \end{aligned}$$

and
$$F(x, y) = \lim_{(m,\, n) \to \infty} f(x, y, m, n)$$

where $f(x, y, m, n)$ is such that the repeated and the double limits exist and are equal.

These conditions are sufficient to give a unique value for it, *viz.* $F(x, y)$.

Now equation 2.1 becomes

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\int_0^{\infty}\int_0^{\infty} \phi_{mn}(x, y)\, dxdy\right] = \int_0^{\infty}\int_0^{\infty} F(x, y)\, dxdy. \qquad 2.2$$

In this equation, the double integral on the right may happen to be convergent, whereas the double series on the left may not be so. In that case, we agree to take the integral as defining the sum of the oscillatory series on the left.

[Here, conditions on $F(x, y)$ must be imposed in order that the order of integration may be changed, without altering its value. The conditions referred to are that $\int_0^{\infty} F(xy)\, dx$, $\int_0^{\infty} F(xy)\, dy$ and either

$$\int_0^{\infty} dx \int_0^{\infty} F(xy)\, dy, \text{ or } \int_0^{\infty} dy \int_0^{\infty} F(xy)\, dx$$

must be convergent.]

The special type of ϕ's that will be taken now is

$$\phi_{m,n}(x, y) = e^{-x-y} \frac{u_{m,n} x^m y^n}{m!\, n!}$$

where u_{mn} is independent of both x and y.

Now we see that

$$\int_0^\infty\int_0^\infty \phi_{m,n}\, dxdy = \int_0^\infty\int_0^\infty e^{-x-y}\frac{u_{m,n}x^m y^n}{m!\; n!}\, dxdy = u_{m,n} \qquad 2.3$$

$$\text{and } F(x, y) = e^{-x-y}\left|\begin{array}{l} u_{00} + u_{01}y + u_{02}\frac{y^2}{2!} + \dots\dots + u_{0n}\frac{y^n}{n!} + \dots\dots \\ + u_{10}x + \dots\dots\dots\dots \\ + \dots\dots\dots\dots \\ + u_{m0}\frac{x^m}{m!} + \dots\dots\dots + \frac{u_{mn}x^m y^n}{m!\; n!} \dots\dots \end{array}\right|$$

$$= e^{-x-y}\, u(x, y). \dots \quad \dots \quad \dots \quad \dots \qquad 2.4$$

It is to be assumed that the co-efficients $u_{m,n}$ are such that the double series is convergent for all values of x and y in Pringsheim's sense; that is, the rows and the columns converge and the double series is convergent.

Then, provided that $\int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dxdy$ is convergent, we may agree to associate its value with the double series $\sum_0^\infty \sum_0^\infty u_{m,n}$ if this is non-convergent, and the integral may be called the *sum* of the series. Let the sum be denoted by the symbol $\overset{\infty}{\underset{0}{S}}\,\overset{\infty}{\underset{0}{S}}\,(u_{m,n})$.

3. CONDITION OF CONSISTENCY: Our definition will be an absurdity unless $S\,S\,(u_{m,n})$ agrees with $\Sigma\Sigma\, u_{m,n}$ whenever the latter converges to a *definite* sum, that is, if $\Sigma\Sigma\, u_{mn}$ converges to a definite sum s then

$$\int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y) dxdy$$

must have the same value s.

Since $u_{m,n}$ is convergent and satisfies Pringsheim's conditions, we can find numbers μ and ν such that

$$| u_{m,n} | < \varepsilon, \text{ if } m > \mu \text{ and } n > \nu.$$

Then
$$\left| \sum_m^\infty \sum_n^\infty u_{mn} e^{-x-y} \frac{x^m y^n}{m!\, n!} \right| < \varepsilon \left| e^{-x-y} \sum_m^\infty \sum_n^\infty \frac{x^m y^n}{m!\, n!} \right|$$
$$< \varepsilon, \text{ if } m > \mu \text{ and } n > \nu. \qquad 3.1$$

Therefore the double series converges uniformly for all values of x and y within the intervals (o, p), (o, q), where p and q may be arbitrarily large.

$$\therefore \int_0^p\int_0^q e^{-x-y}\, u(x,y)dxdy = \sum_0^\infty\sum_0^\infty u_{m,n}\int_0^p\int_0^q e^{-x-y}\,\frac{x^m y^n}{m!\, n!}\, dxdy.^* \qquad 3.2$$

$$= \sum_0^\infty\sum_0^\infty u_{mn}\, p_m q_n \quad \ldots \quad \ldots \quad 3.3$$

where
$$p_m q_n = \int_0^p\int_0^q \frac{e^{-x-y}\, x^m y^n}{m!\, n!}\, dxdy$$

$$= e^{-p-q}\left(\frac{p^{m+1}}{(m+1)!} + \frac{p^{m+2}}{(m+2)!}\ldots\right)\left(\frac{q^{n+1}}{(n+1)!} + \ldots\right) \quad 3.4$$

It is evident that $p_m q_n$ is a decreasing sequence and can be arranged as a double series

$$e^{-p-q}\left|\begin{array}{l} \dfrac{p^{m+1}q^{n+1}}{(m+1)!\,(n+1)!} + \dfrac{p^{m+1}q^{n+2}}{(m+1)!\,(n+2)!} + \ldots\ldots\ldots \\ + \dfrac{p^{m+2}q^{n+1}}{(m+2)!\,(n+1)!} + \dfrac{p^{m+2}q^{n+2}}{(m+2)!\,(n+2)!} + \ldots\ldots\ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right| \quad \ldots \quad 3.5$$

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* Since
$$\Sigma\Sigma\, u_{mn}\, e^{-x-y}\,\frac{x^m y^n}{m!\, n!}$$
is uniformly convergent, μ and ν can be chosen such that
$$\sum_m^\infty\sum_n^\infty u_{m,n}\, e^{-x-y}\,\frac{x^m y^n}{m!\, n!} < \varepsilon, \text{ whenever } m > \mu \text{ and } n > \nu.$$
Hence we have
$$\left|\sum_m^\infty\sum_n^\infty\int_0^p\int_0^q u_{mn}\, e^{-x-y}\,\frac{x^m y^n}{m!\, n!}\, dxdy\right| < \varepsilon\, pq.$$
At the same time we have
$$\left| e^{-x-y}\, u(x,y) - \sum_0^{m-1}\sum_0^{n-1} u_{m,n}\, e^{-x-y}\,\frac{x^m y^n}{m!\, n!}\right| < \varepsilon$$
$$\therefore \left|\int_0^p\int_0^q e^{-x-y}\, u(xy) - \int_0^p\int_0^q\sum_0^{m-1}\sum_0^{n-1} u_{mn}\,\frac{e^{-x-y}x^m y^n}{m!\, n!}\right| < \varepsilon\, pq.$$
i.e.,
$$\left|\int_0^p\int_0^q e^{-x-y}\, u(x,y) - \sum_0^{m-1}\sum_0^{n-1} u_{m,n}\int_0^p\int_0^q\frac{e^{-x-y}x^m y^n}{m!\, n!}\right| < \varepsilon\, pq.$$
$$\therefore \left|\int_0^p\int_0^q e^{-x-y}\, u(x,y) - \sum_0^\infty\sum_0^\infty u_{mn}\int_0^p\int_0^q\frac{e^{-x-y}x^m y^n}{m!\, n!}\right| < 2\varepsilon\, pq.$$

Since p and q are positive and the double sequence $p_m q_n$ is never increasing such that

$$p_v q_\mu - p_{v+1} q_\mu, \; p_v q_\mu - p_v q_{\mu+1},$$

and
$$p_v q_\mu - p_v q_{\mu+1} - p_{v+1} q_\mu + p_{v+1} q_{\mu+1},$$

are positive, applying the extended form of Abel's lemma for double series, we have

$$h(p_0 q_0 - p_m q_n) + h_{mn} p_m q_n$$
$$< \sum_0^\infty \sum_0^\infty u_{mn} p_m q_n < H(p_0 q_0 - p_m q_n) + H_{mn} p_m q_n. \quad \dots \quad 3.6$$

where h and H are the lower and the upper limits of $\sum_0^x \sum_0^y u_{xy}$ as x varies from 0 to $m-1$ and y varies from 0 to $n-1$; h_{mn} and H_{mn} are the lower and upper limits of the same as x varies from m to ∞ and y from n to ∞.

Now if $\Sigma \Sigma u_{mn}$ converges to the sum s in Pringsheim's sense, we can find m and n corresponding to a given ε such that $h_{mn} \geqslant s - \varepsilon$ and $H_{mn} \leqslant s + \varepsilon$ however small ε be.

As p and q tend to ∞,

$$\lim. p_0 q_0 = 1 \text{ and } \lim. p_m q_n = 1.$$

For,

$$p_0 q_0 \, e^{p+q} = pq + \frac{pq^2}{1.2!} + \frac{pq^3}{1.3!} + \dots\dots$$
$$+ \frac{p^2 q}{2!\,1!} + \frac{p^2 q^2}{2!\,2!} + \dots\dots\dots\dots$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$= (e^p - 1)(e^q - 1).$$

$$\therefore \; \lim. p_0 q_0 = \lim_{p,q\to\infty} (e^p - 1)(e^q - 1)\, e^{-p-q} = 1.$$

Similarly

$$\lim. p_m q_n = \lim. e^{-p-q} \left(e^p - 1 - \frac{p}{1} - \frac{p^2}{2!} - - \frac{p^{m-1}}{m-1!}\right) \times \left(e^q - 1 - \frac{q}{1} \dots\dots\dots - \frac{q^{n-1}}{n-1!}\right) = 1]$$

Hence $\lim_{p,q\to\infty} h(p_0 q_0 - p_m q_n) + h_{mn} p_m q_n \geqslant s - \varepsilon$

and $\lim_{p,q\to\infty} H(p_0 q_0 - p_m q_n) + H_{mn} p_m q_n \leqslant s + \varepsilon.$

$$\therefore \quad s-\varepsilon \leqslant \lim_{p,\, q \to \infty} \Sigma\Sigma u_{mn}\, p_m\, q_n \leqslant s+\varepsilon.$$

But ε being arbitrary, the max. and min. limits are equal.

$$\therefore \quad \lim_{p,\, q \to \infty} \sum_0^\infty \sum_0^\infty u_{mn}\, p_m\, q_n = s.$$

But, if $\Sigma\Sigma u_{mn}$ diverges, say to $+\infty$, we can find m and n such that $h_{mn} > \text{N}$, however great N may be.

$$\therefore \quad \lim_{p,\, q \to \infty} h\,(p_0\, q_0 - p_m\, q_n) + h_{mn}\, p_m\, q_n = h_{mn} > \text{N}.$$

$$\therefore \quad \lim_{p,\, q \to \infty} \Sigma\Sigma u_{mn}\, p_m\, q_n > \text{N}.$$

Since N is arbitrary, this implies that $\Sigma\Sigma u_{mn}\, p_m q_n$ tends to infinity with p and q.

4. Let us apply the theory developed above to find the sum of certain oscillatory double series.

(1) Let
$$\begin{aligned} S = \;& 1-1+1-1+\ldots\ldots \\ & -1+1-1+1\ldots\ldots\ldots \\ & +1-1+1-1\ldots\ldots\ldots \\ & -1+1-1+1\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\begin{aligned} \text{Now } u\,(x, y) = \;& 1-\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots\ldots\ldots \\ & -\frac{y}{1!}+\frac{xy}{1!\,1!}-\frac{x^2y}{2!\,1!}\ldots\ldots \\ & +\frac{y^2}{2!}-\frac{xy^2}{1!\,2!}+\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ = \;& e^{-x-y}. \end{aligned}$$

$$\begin{aligned} \therefore \quad S &= \int_0^\infty\!\!\int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy \\ &= \int_0^\infty\!\!\int_0^\infty e^{-2x-2y}\, dxdy = \tfrac{1}{4}. \end{aligned}$$

(2) Again, let
$$\begin{aligned} S = \;& 1-1+1-1+\ldots\ldots\ldots\ldots \\ & -2+2-2+2-\ldots\ldots\ldots\ldots \\ & +2^2-2^2+2^2-2^2\ \ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Here $u(x, y) = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$

$$- \frac{2y}{1!} + \frac{2xy}{1!\,1!} - \frac{2x^2y}{2!\,1!} + \frac{2x^3y}{3!\,1!} - \ldots$$

$$+ \frac{2^2y^2}{2!} - \frac{2xy^2}{1!\,2!} + \frac{2x^2y^2}{2!\,2!} - \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= e^{-x-2y}.$$

$$\therefore \quad S = \int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dxdy = \int_0^\infty \int_0^\infty e^{-2x-3y}\, dxdy = \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{1}{6}.$$

(3) If $S = 1 - 2 + 3 - 4 + \ldots\ldots$

$$- 2 + 3 - 4 + 5 - \ldots\ldots$$

$$+ 3 - 4 + 5 - \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

then $\quad u(x, y) = e^{-x-y}(1 - x - y).$

$$\therefore \quad S = \int_0^\infty \int_0^\infty e^{-2x-2y}(1 - x - y)\, dxdy = 0.$$

5. SUMMABILITY BY ROWS AND BY COLUMNS:

Let us consider the $(r + 1)$th row of the double series $\overset{\infty}{\underset{0}{S}}\overset{\infty}{\underset{0}{S}}(u_{mn})$. It is

$$u_{r,0} + u_{r,1} + u_{r,2} + u_{r,3} + \ldots\ldots$$

which can be represented by the symbol $\overset{n=\infty}{\underset{n=0}{S}}(u_{rn})$, provided the row is summable.* The sum of the double series can now be taken as

$$\overset{\infty}{\underset{r=0}{S}}\left[\overset{\infty}{\underset{n=0}{S}} u_{rn}\right]$$

provided $\quad v_0 + v_1 + v_2 + \ldots\ldots\ldots\ldots$

is summable, where

$$v_r = \overset{\infty}{\underset{n=0}{S}}(u_{rn}).$$

We get a similar result by considering the columns.

It can be seen that the result is

$$\overset{\infty}{\underset{n=0}{S}}\left[\overset{\infty}{\underset{r=0}{S}} u_{rn}\right]$$

provided that the columns and the sums of the columns are summable. Now the question arises whether

$$\overset{\infty}{\underset{r=0}{S}}\left[\overset{\infty}{\underset{n=0}{S}} u_{rn}\right] \text{ and } \overset{\infty}{\underset{n=0}{S}}\left[\overset{\infty}{\underset{r=0}{S}} u_{rn}\right]$$

* What is meant by summability here, is that, even though the series is oscillatory or divergent, the corresponding integral is convergent.

are equal to each other. It can be verified that this is true for the examples dealt with in § 4 of this paper.

To prove that this is true generally, let us take the row corresponding to $\overset{\infty}{\underset{n=0}{S}} (u_{rn})$ in $F(x, y)$. It is

$$e^{-x-y} \frac{x^r}{r!} \left(u_{r,0} + u_{r,1} \frac{y}{1!} + u_{r,2} \frac{y^2}{2!} + \ldots\ldots \right) \qquad 5.1$$

$$= e^{-x-y} \frac{x^r}{r!} \cdot U_r(y), \text{ (say)}.$$

Since we have assumed $\overset{\infty}{\underset{n=0}{S}} (u_{rn})$ to be summable,

$$\left(\int_0^\infty e^{-y} U_r(y)\, dy\right)$$

is convergent.

The sum for the $(r + 1)$th row is given by the integral

$$e^{-x} \frac{x^r}{r!} \int_0^\infty e^{-y} U_r(y) dy.$$

Hence the sum of the double series by rows is

$$\int_0^\infty \left[e^{-x} \frac{x^r}{r!} \int_0^\infty e^{-y} U_r(y) dy \right] dx$$

and its sum by columns is

$$\int_0^\infty \left[e^{-y} \frac{y^n}{n!} \int_0^\infty e^{-x} U_n(x) dx \right] dy.$$

Since we have assumed u_{mn} to be such that the double series

$$\sum_0^\infty \sum_0^\infty u_{mn} x^m y^n$$

converges, and converges by rows as well as by columns, the double series converges to the same sum, whether taken by rows or by columns.

$$\therefore \qquad \Sigma \frac{x^r}{r!} U_r(y) = \Sigma \frac{y^n}{n!} U_n(x) = u(x, y).$$

Hence the two double integrals for the sum of the summable series are

$$\int_0^\infty e^{-x} dx \int_0^\infty e^{-y} u(x, y) dy \text{ and } \int_0^\infty e^{-y} dy \int_0^\infty e^{-x} u(x, y) dx$$

which can be written also as

$$\int_0^\infty dx \int_0^\infty e^{-x-y} u(x, y) dy \text{ and } \int_0^\infty dy \int_0^\infty e^{-x-y} u(x, y) dx.$$

Since the series is summable by rows

$$\int_0^\infty e^{-x-y}\, u(x, y) dy$$

must converge, and since it is summable by columns

$$\int_0^\infty e^{-x-y}\, u(x, y) dx$$

must converge, and since the double series itself is summable, the corresponding double integral

$$\int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy$$

is convergent. And de la Vallee Poussin has shown that, if the three integrals mentioned above are convergent, then

$$\int_0^\infty dx \int_0^\infty e^{-x-y}\, u\,(x, y)\, dy = \int_0^\infty dy \int_0^\infty e^{-x-y}\, u\,(x, y)\, dx.$$

Hence for an oscillatory summable double series summable* by rows and by columns, sum of the double series is equal to the sum obtained, either by rows or by columns.

In concluding, it is interesting to note the intimate relations between a double summable series and the corresponding double integral. Let us take the series $\overset{\infty}{\underset{0}{S}}\, \overset{\infty}{\underset{0}{S}}\, u_{mn}$. This is summable by rows and the results obtained, thus are also summable. Considering the corresponding double integral $\int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy$, de la Vallee Poussin's condition that $\int_0^\infty e^{-x-y}\, u\,(x, y)\, dy$ should converge, implies the property of summability of the sums of the rows. The convergence of $\int_0^\infty e^{-x-y}\, u\,(x, y)\, dx$ implies similarly the property of summability of the sums of the columns. Thus we see that, if a *summable double series were to have a unique sum, when taken by rows or columns, or as a double series, it is necessary that the corresponding double integral should satisfy the conditions of De La Vallee Poussin in virtue of which the value of the double integral is the same, even though there is a change of order of integration.*

(*To be continued.*)

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* 'Summable' is understood in the restricted sense of § 2 of this paper. Such series as $1 + 2 + 3 + \ldots\ldots\ldots\ldots$ are not contemplated in connection with this paper.

On Some Infinite Products and Series: Part II.

By M. BHIMASEÑA RAO AND M. VEÑKATARAMA AYYAR.

[In Part I* of this paper several general results were obtained wherein a was put equal to $\pi r(\cos\theta + i\sin\theta)$ and real and imaginary parts were separated. The equated *real* parts led to the evaluation of several *theta*-products and a few series. In this part, we propose to deal with the results obtained on equating the *imaginary* parts. We add also a *direct* method of establishing a general type of relation between several of the series got. In the course of that work, a number of other results are found.]

§ 1. In result I of Part I, put

$$a = \pi r(\cos\theta + i\sin\theta),$$

and equate the imaginary parts. We get

$$-\left[\frac{e^{2\pi r\cos\theta}\sin(2\pi r\sin\theta)}{e^{4\pi r\cos\theta} - 2e^{2\pi r\cos\theta}\cos(2\pi r\sin\theta) + 1} + 2^{2n+1}\frac{e^{4\pi r\cos\theta}\sin(4\pi r\sin\theta)}{e^{8\pi r\cos\theta} - 2e^{4\pi r\cos\theta}\cos(4\pi r\sin\theta) + 1} + \ldots\ldots\right]$$

$$+(-1)^n\left[\frac{\cos(2n+2)\theta}{r^{2n+2}}\right]\times$$

$$\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\sin\left(\frac{2\pi}{r}\sin\theta\right)}{e^{\frac{4\pi}{r}\cos\theta} - 2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right) + 1} + 2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\sin\left(\frac{4\pi}{r}\sin\theta\right)}{e^{\frac{8\pi}{r}\cos\theta} - 2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right) + 1} + \ldots\ldots\right\}$$

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* *Vide*: *J. I. M.* S., Vol. XV., No. 7, pp. 150—162.

$$+ (-1)^{n+1} \left[\frac{\sin(2n+2)\theta}{r^{2n+2}}\right] \times$$

$$\left\{ \frac{e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right) - 1}{e^{\frac{4\pi}{r}\cos\theta} - 2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right) + 1} \right.$$

$$\left. + 2^{2n+1} \frac{e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right) - 1}{e^{\frac{8\pi}{r}\cos\theta} - 2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right) + 1} + \ldots\ldots \right\}$$

$$= \frac{B_{n+1}}{4(n+1)} \cdot \frac{-\sin(2n+2)\theta}{r^{2n+2}} \quad \ldots \qquad \ldots \text{(A)}$$

In (A), put $r = 1$ and $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$. Then,

$$(-1)^{n+1} \sin(n+1)\frac{\pi}{3} \cdot \left[\frac{-e^{\pi\sqrt{3}} - 1}{(e^{\pi\sqrt{3}} + 1)^2} + 2^{2n+1} \cdot \frac{e^{2\pi\sqrt{3}} - 1}{(e^{2\pi\sqrt{3}} - 1)^2} + \ldots\ldots\right]$$

$$= \frac{B_{n+1}}{4(n+1)} \cdot \left(-\sin(n+1)\frac{\pi}{3}\right).$$

So, when $\sin(n+1)\frac{\pi}{3}$ is not zero, *i.e., when* $n \not\equiv 2 \pmod{3}$,

we get

$$\frac{1^{2n+1}}{1 + e^{\pi\sqrt{3}}} + \frac{2^{2n+1}}{1 - e^{2\pi\sqrt{3}}} + \frac{3^{2n+1}}{1 + e^{3\pi\sqrt{3}}} + \frac{4^{2n+1}}{1 - e^{4\pi\sqrt{3}}} + \ldots\ldots$$

$$= (-1)^{n+1} \cdot \frac{B_{n+1}}{4(n+1)}.^{*} \quad \ldots \qquad \ldots \text{(I)}$$

Again, in (1), put $r = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$. Then,

$$(-1)^{n+1} 2^{n+1} \sin(n+1)\frac{\pi}{2} \left[\frac{e^{2\pi} - 1}{(e^{2\pi} - 1)^2} + 2^{2n+1} \cdot \frac{e^{4\pi} - 1}{(e^{4\pi} - 1)^2} + \ldots\ldots\right]$$

$$= 2^{n+1} \frac{B_{n+1}}{4(n+1)} \cdot \left(-\sin(n+1)\frac{\pi}{2}\right)$$

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* Compare with result XVIII of Part I.

When $\sin(n+1)\frac{\pi}{2}$ is not zero, *i.e.*, when *n is even*,

$$\frac{1}{e^{2\pi}-1}+\frac{2^{2n+1}}{e^{4\pi}-1}+\frac{3^{2n+1}}{e^{6\pi}-1}+\ldots\ldots\ldots\ldots$$
$$=\frac{B_{n+1}}{4(n+1)}.\dagger \quad \ldots \qquad \ldots \text{(II-}a\text{)}$$

Once again, in (A), put $r=\surd 2$, and $\sin\theta=\frac{1}{\surd 2}$. Then,

$$(-1)^{n+1}\frac{\sin(n+1)\frac{\pi}{2}}{2^{n+1}}\cdot\left[\frac{-e^{\pi}-1}{(e^{\pi}+1)^2}+2^{2n+1}\cdot\frac{e^{2\pi}-1}{(e^{2\pi}-1)^2}+\ldots\ldots\right]$$
$$=\frac{B_{n+1}}{4(n+1)}\cdot\left\{-\frac{\sin(n+1)\frac{\pi}{2}}{2_{n}+1}\right\}.$$

Here also, when *n is even*,

$$\frac{1}{1+e^{\pi}}+\frac{2^{2n+1}}{1-e^{2\pi}}+\frac{3^{2n+1}}{1+e^{3\pi}}+\frac{4^{2n+1}}{1-e^{4\pi}}+\ldots\ldots$$
$$=-\frac{B_{n+1}}{4(n+1)}. \quad \ldots \qquad \ldots \text{(II-}b\text{)}$$

As in both cases, *n is even*, adding (II-*a*) and (II-*b*), we have

$$\operatorname{cosech}(\pi)-2^{2n+1}\operatorname{cosech}(2\pi)+3^{2n+1}\operatorname{cosech}(3\pi)$$
$$-4^{2n+1}\operatorname{cosech}(4\pi)+\ldots\ldots\ldots\ldots=0. \qquad \ldots \text{(II)}$$

Again, in (A), put $r=\frac{1}{2}$ and $\sin\theta=\frac{1}{2}$, $\cos\theta=\frac{\surd 3}{2}$. Then,

$$-\left\{\frac{e^{\frac{\pi\surd 3}{2}}}{e^{\pi\surd 3}+1}+3^{2n+1}\cdot\frac{-e^{\frac{3\pi\surd 3}{2}}}{e^{3\pi\surd 3}+1}+5^{2n+1}\cdot\frac{e^{\frac{5\pi\surd 3}{2}}}{e^{5\pi\surd 3}+1}+\ldots\ldots\right\}$$
$$+(-1)^{n+1}\sin(n+1)\frac{\pi}{3}\;2^{2n+2}\times$$
$$\left[\frac{e^{2\pi\surd 3}-1}{(e^{2\pi\surd 3}-1)^2}+2^{2n+1}\cdot\frac{e^{4\pi\surd 3}-1}{(e^{4\pi\surd 3}-1)^2}+\ldots\ldots\right]$$
$$=\frac{-B_{n+1}}{4(n+1)}\cdot\sin(n+1)\frac{\pi}{3}\cdot 2^{2n+2}.$$

† Compare with result II of Part I.

Here, let $n \equiv 2 \pmod 3$, making the 2nd part of the L. H. S. and the R. H. S. zero. Then, we get

$$\operatorname{sech}\left(\frac{\pi\sqrt{3}}{2}\right) - 3^{2n+1}\operatorname{sech}\left(\frac{3\pi\sqrt{3}}{2}\right) + 5^{2n+1}\operatorname{sech}\left(\frac{5\pi\sqrt{3}}{2}\right) - \ldots\ldots = 0, \qquad \ldots \text{(III)}$$

a result parallel to a problem of the late S. Ramanujan, F. R. S., which occurs as result IV below.

§ 2. Take result IV of Part I and therein put

$$a = \pi r(\cos\theta + i\sin\theta),$$

and equate the imaginary parts. Then,

$$-\left[\frac{e^{\pi r\cos\theta}\sin(\pi r\sin\theta)}{1+2e^{\pi r\cos\theta}\cos(\pi r\sin\theta)+e^{2\pi r\cos\theta}} + 3^{2n+1}\cdot\frac{e^{3\pi r\cos\theta}\sin(3\pi r\sin\theta)}{1+2e^{3\pi r\cos\theta}\cos(3\pi r\sin\theta)+e^{6\pi r\sin\theta}}+\ldots\ldots\right]$$

$$+(-1)^n\frac{\cos(2n+2)\theta}{r^{2n+2}}\cdot\left\{\frac{e^{\frac{\pi}{r}\cos\theta}\sin\left(\frac{\pi}{r}\sin\theta\right)}{1+2e^{\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)+e^{\frac{2\pi}{r}\cos\theta}} + 3^{2n+1}\cdot\frac{e^{\frac{3\pi}{r}\cos\theta}\sin\left(\frac{3\pi}{r}\sin\theta\right)}{1+2e^{\frac{3\pi}{r}\cos\theta}\cos\left(\frac{3\pi}{r}\sin\theta\right)+e^{\frac{6\pi}{r}\cos\theta}}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\cdot\frac{\sin(2n+2)\theta}{r^{2n+2}}\cdot\left\{\frac{1+e^{\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)}{1+2e^{\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)+e^{\frac{2\pi}{r}\cos\theta}} + 3^{2n+1}\frac{1+e^{\frac{3\pi}{r}\cos\theta}\cos\left(\frac{3\pi}{r}\sin\theta\right)}{1+2e^{\frac{3\pi}{r}\cos\theta}\cos\left(\frac{3\pi}{r}\sin\theta\right)+e^{\frac{6\pi}{r}\cos\theta}}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot(2^{2n+1}-1)\cdot\left(\frac{-\sin(2n+2)\theta}{r^{2n+2}}\right). \qquad \ldots \text{(B)}$$

Notes and Questions.

Did the Greeks Solve the Quadratic Equation?

One frequently meets with the statement that the ancient Greeks solved the quadratic equation. For instance, on page 250 of Volume I of Pascal's *Repertorium der höheren Mathematik*, 1910, it is stated that the solution of the quadratic equation was known to the Greeks. On page 126 of Volume I of Smith's *History of Mathematics*, 1923, we find the stronger statement "that the general quadratic as we know it to-day was thus fully mastered by the Greek mathematicians." It seems to the writer of the present note that it would be more correct to say that the ancient Greeks found formulas by means of which the quadratic equation can be solved, but they did not advance far enough in their algebraic work to solve the general quadratic equation in one unknown. By means of the formulas they found one root of many special quadratic equations.

While the expression 'solution of the algebraic equation of the second degree' is not very definite, the student naturally understands by it at least as much knowledge about this equation as is expected of him when he is examined on this equation in a course in elementary algebra. For instance, one fundamental fact about this equation is that it always has a root. Since the Greeks knew nothing about imaginary numbers they could not have known this fact. Another fundamental fact is that the first member of the equation $ax^2 + bx + c = 0$ can always be resolved into two linear factors. As the Greeks were ignorant of the factor theorem, even in the elementary form as applied to the general quadratic expression in one unknown, they could not have known this fact even in the case when the two roots are positive and rational. As the Greeks knew nothing about equal roots they, of course, did not know anything about a general criterion relating to the equality of the roots of the quadratic equation. They also knew nothing about the relations between the co-efficients and the roots of this equation.

It might be said that when the student comes across the statement that the Greeks 'mastered the quadratic equation,' he should not be misled thereby since people who did not know even *negative* numbers could not have 'mastered' the quadratic equation in the strict modern sense. There is no doubt, however, that the above statement has given rise to much confusion on the part of the reader. Moreover the fact that the Greeks

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found general formulas for the solution of the quadratic equation, which were not fully understood by them or by other mathematicians for a period of one thousand years after their discovery, gives some insight into the extremely slow historical development of the subject. More than two thousand years may have passed from the time of this discovery until imaginary numbers were fully accepted as legitimate numbers through the labours of Caspar Wessel, J. R. Argand, C. F. Gauss, W. R. Hamilton, and many others.

The great difference between the casual discovery of a formula and the true understanding of the full significance of the formula is perhaps nowhere more clearly exhibited than in the history of the solution of the quadratic equation. Hence it seems desirable that the student's attention should be directed to this point. He will thereby learn a very fundamental fact in the history of mathematics. For instance, he will see that it is not quite accurate to say that the general cubic equation was solved algebraically by the Italian mathematicians during the first half of the sixteenth century, though it is true that a formula, which is now commonly called Cardan's formula, was discovered during this period. More than two hundred years passed from the time when this formula appeared in the *Ars Magna* (1545), before it was pointed out by D'Alembert in the *Encyclopédie ou dictionnaire raisonnée des sciences*, Volume 2, page 736, that this formula represents not only one root but the three roots of the cubic equation.

Just as in the case of the quadratic equation, the formula for the general solution of the algebraic equation of the third degree was discovered a long time before this formula was fully understood. In both cases, the formulas were first used to find *one root* of special equations. In both cases, those who discovered them knew comparatively little about their general significance. It is true that each of them took a very important step towards the solution of the equation in question, but it was only a step. This step had to be followed by the work of many others before the 'solution'—in the modern sense of this term—was completed. A knowledge of the fact that the labours of many mathematicians, who lived at different times and in different countries, are reflected in some of the most fundamental results with which the student of elementary mathematics has to deal should help him to realize and appreciate the value of modern co-operative efforts as exhibited by the modern mathematical societies and their periodicals.

From what precedes it is easy to infer that our answer to the question which is the subject of the present note is that the Greeks did not

solve the quadratic equation. We would add that the Greeks, the Indians and the Arabs, taken together, did not solve this equation, although each of them made important contributions towards the solution. In fact, contributions towards the solution as it is now presented to students of elementary mathematics extend into the nineteenth century and include the work of those who laid a solid foundation for the use of the negative and the general complex numbers. The more advanced student will find that the modern theory of irrational numbers and the Galois theory of equations throw additional light on this solution, though the term solution need not imply such deep insight. It seems, however, clear that those who considered many quadratic equations as *insolvable* should not be regarded as having 'solved' the general quadratic equation, unless the term *solved* is used in a restricted sense.

G. A. MILLER.

On the Eight Spheres touching the faces of a Tetrahedron.

With the usual convention of signs, the centres of the eight spheres are the points $(1, \pm 1, \pm 1, \pm 1)$. Denote these by $I, I_1, I_2, I_3, I_4, I_a, I_b, I_c$. Let the corresponding radii be $r, r_1, r_2, r_3, r_4, r_a, r_b, r_c$.

We have

$$\text{vol. IBCD} + \text{vol. ICDA} + \text{vol. IDAB} + \text{vol. IABC} = \text{vol. ABCD}.$$

$$\therefore\ r(A_1 + A_2 + A_3 + A_4) = 3 \text{ vol. ABCD}.$$

$$\therefore\ \frac{1}{r} = \frac{2}{3} \cdot \frac{S}{V} \qquad \ldots \qquad \ldots \quad (1)$$

where A_1, A_2, A_3, A_4 are the areas of faces opposite to A, B, C, D and

$$2S = A_1 + A_2 + A_3 + A_4,$$

and

$$V = \text{vol. ABCD}.$$

Similarly $$r_1^{-1} = \frac{2}{3} \cdot \frac{(S - A_1)}{V},\ r_2^{-1} = \frac{2}{3} \cdot \frac{(S - A_2)}{V}, \text{ etc.} \qquad \ldots (1.1)$$

and $$r_a^{-1} = \frac{2}{3} \cdot \frac{(S - A_1 - A_2)}{V}, \text{ etc.} \qquad \ldots (1.2)$$

From relations (1.1), (1.2), we deduce

$$2 \cdot r^{-1} = r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1}; \qquad \ldots (1.3)$$

$$2 \cdot r_a^{-1} = r_1^{-1} + r_2^{-1} - r_3^{-1} - r_4^{-1}, \text{ \&c.} \qquad \ldots (1.4)$$

2. Let the spheres touch the face BCD opposite to A in α, α_1, α_2, α_3, α_4, α_a, α_b, α_c; the face opposite to B in β, β_1, β_2, ... β_c; and so on. Further, let the plane angles at the vertices A, B, C, D be denoted by letters A, B, C, D with suffixes 1, 2, 3, 4, according as they lie in the faces opposite A, B, C, D. Thus, angle BAD is A_3. With this notation, we easily see that triangles $A\beta C$ and $A\delta C$ are congruent.

$$\therefore \quad C\hat{A}\delta = C\hat{A}\beta.$$

Similarly $\quad D\hat{A}\beta = D\hat{A}\gamma,$

$$\therefore \quad B\hat{A}\gamma = B\hat{A}\delta.$$

Hence $\quad C\hat{A}\beta = C\hat{A}\delta = (\sigma_1 - A_3)$, etc.

Similarly $\quad D\hat{A}\beta_1 = \sigma_1 - A_3$, etc.

Thus it will be seen that β, β_1, *are isogonal conjugates with respect to the face ACD.* Similarly the six points β_2, β_3, β_4, β_a, β_b, β_c can be divided into pairs of isogonal conjugate points. Thus (β, β_1), (β_2, β_a), (β_3, β_b), (β_4, β_c), are isogonal pairs for the triangle ACD.

Corresponding properties hold for the remaining faces.

3. The perpendicular IP on CD is perpendicular to the plane I_1CD; for, the plane ICD is perpendicular to the plane I_1CD.

$\therefore$ IP is perpendicular to I_1P.

Thus II_1 subtends a right angle at the six feet of perpendiculars from I, I_1 on BC, CD, DB and so they lie on a circle.

Similarly, it can be proved that the 24 feet of the perpendiculars from the centres of the eight spheres on BC, CD, DB lie six by six on four circles.

Further, since IP is perpendicular to CD and also $I\alpha$ is perpendicular to the plane BCD, αP is perpendicular to CD. Thus the feet of the perpendiculars above coincide with those from α, α_1, α_2, ... α_a.....α_c on the sides of the triangle BCD.

4. From § 2 we see that (α, α_1), (α_2, α_a), (α_3, α_b), (α_4, α_c) are the foci of conics touching the sides of ΔBCD; and that the circles of § 3 are the auxiliary circles of these conics.

If X is the foot of the $\perp^r$ from A on BCD, then since A, I, I_1 are collinear, their projections X, α, α_1, are collinear. Similarly, the joins of $\alpha_2\alpha_a$, $\alpha_3\alpha_b$, $\alpha_4\alpha_c$ pass through the same point X.

5. Since the planes IBC, I_1BC, bisect the angle between the planes DBC, ABC,

$$\{AP_1, II_1\} = -1,$$

P_1 being the point where II_1 meets the plane BCD. But

$$\{XP_1, \alpha\alpha_1\} = \{AP_1, II_1\} = -1.$$

Similarly if Q_1, R_1, S_1 are the points where I_2I_a, I_3I_b, I_4I_c meet the plane BCD, then Q_1, R_1, S_1 are the harmonic conjugates of X with respect to I_2I_a, etc.

It can be proved that P_1, Q_1, R_1, S_1 are the vertices of a quadrangle of which the triangle ABC is the diagonal triangle.

6. The following are some properties of the tetrahedron in which opposite edges are equal :—

(1) Of the eight spheres, only *five* are finite.

(2) Their radii are given by

$$2r = r_1 = r_2 = r_3 = r_4 = \frac{1}{\sqrt{2}}\left\{\frac{1}{d^2-a^2} + \frac{1}{d^2-b^2} + \frac{1}{d^2-c^2}\right\}^{-\frac{1}{2}}\Big]$$

where d is the diameter of the circumsphere and a, b, c are the edges.

(3) The points α_2, α_3, α_4, β_1, β_3, β_4, γ_1, γ_2, γ_4, δ_1, δ_2, δ_3, all lie on the circumsphere.

(4) $II_1 = II_2 = II_3 = II_4 = R$.

(5) The inscribed spheres of the tetrahedra ABCD, $I_1I_2I_3I_4$ coincide.

C. SALDANHA AND B. B. BAGI.

Solutions.

Question 1257.

(A. C. L. WILKINSON):—Solve the partial differential equation:

$$r(1-q^2)^2 + 2pqs(1-q^2) + p^2q^2t - (1-q^2)(rt-s^2)z = 0.$$

Solution by Martyn Thomas.

The given equation is of the type $Rr + Ss + Tt + U(rt - s^2) = V$, where $R = (1-q^2)^2$, $S = 2pq(1-q^2)$, $T = p^2q^2$, $U = (q^2-1)z$ and $V = 0$, (discussed fully in Forsyth's *Differential Equation.*)

The auxiliary equation for λ is

$$\lambda^2(RT + UV) + \lambda US + U^2 = 0.$$

$$\therefore\ \lambda^2(1-q^2)^2p^2q^2 + \lambda(q^2-1)z \cdot 2pq(1-q^2) + (q^2-1)^2z^2 = 0.$$

$$\therefore\ (1-q^2)^2\{\lambda pq - z\}^2 = 0,$$

giving two equal roots for λ.

$$\therefore\ \lambda_1 = \lambda_2 = \frac{z}{pq}.$$

An intermediary integral is to be sought by solving the system

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0, \quad \ldots \quad (1)$$

$$Udx + \lambda_2 Rdy + \lambda_2 U\,dq = 0, \quad \ldots \quad (2)$$

$$dz = pdx + qdy. \quad \ldots \quad (3)$$

Now (1) gives

$$-(1-q^2)zdy + \frac{z}{pq}p^2q^2\,dx + \frac{z^2}{pq}(q^2-1)\,dp = 0.$$

$$\therefore\ -pq(1-q^2)\,dy + p^2q^2\,dx - z(1-q^2)\,dp = 0.$$

$$\therefore\ pq^2(pdx + qdy) - pqdy - z(1-q^2)\,dp = 0.$$

$$\therefore\ pq^2 . dz - pq . dy - z(1-q^2)\,dp = 0.$$

i.e.,

$$qdz - dy - \frac{z(1-q^2)}{pq}\,dp = 0. \ldots \quad \ldots \quad (4)$$

(2) gives

$$-(1-q^2)z . dx + \frac{z}{pq}(1-q^2)^2\,dy + \frac{z}{pq}(q^2-1)z\,dq = 0.$$

$\therefore -pq\,.dx + (1-q^2)\,dy - zdq = 0;\ 1-q^2 = 0$ giving $q = \pm 1$,

so that $$z = \pm y + \phi(x) + \text{const.}$$

$$\therefore -q\,(pdx + qdy) + dy - zdq = 0.$$

i.e., $$-qdz + dy - zdq = 0. \quad \ldots \quad \ldots (5)$$

Adding (4) and (5), $$\frac{z\,(1-q^2)}{pq}\,dp + zdq = 0.$$

$$\therefore \quad \frac{qdq}{q^2-1} = \frac{dp}{p}.$$

$$\therefore \quad \tfrac{1}{2}\log(q^2-1) = \log p + \text{const.}$$

$$\therefore \quad p = k\,(q^2-1)^{\frac{1}{2}} \quad \ldots \quad \ldots (6)$$

$$\therefore \quad z = ax + by + c,$$

where $$a = k\,(b^2-1)^{\frac{1}{2}},$$

constants being replaced by functions of $-z \pm y + \phi(x)$.

Question 1263.

(I. Totadri) :—Sum the series :

$$1 - \frac{1-x^m}{1-x} + \frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-x^2)}$$
$$- \frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots\ldots$$

Solution by Hemraj.

Let us write

$$f(x, m) = 1 - \frac{1-x^m}{1-x} + \frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-x^2)} - \ldots\ldots\ldots$$
$$= 1 - (m, 1) + (m, 2) - (m, 3) + \ldots\ldots\ldots$$

Since

$$(m, r) = \frac{[(1-x^{m-r}) + x^{m-r}(1-x^r)]\,(1-x^{m-1})\ldots\ldots\ldots(1-x^{m-r+1})}{(1-x)(1-x^2)\ldots\ldots\ldots(1-x^r)}$$

$$= (m-1, r) + x^{m-r}(m-1, r-1)$$

$$\therefore \quad f(x, m) = (1-x^{m-1}) - (1-x^{m-2})(m-1, 1)$$
$$+ (1-x^{m-3})(m-1, 2) - \ldots\ldots\ldots$$

$$= (1-x^{m-1})\,f(x, m-2) = \ldots\ldots\ldots$$

$$= (1-x^{m-1})(1-x^{m-3})\ldots(1-x^{m-2r+1})\,f(x, m-2r)$$

(*a*) If m is a positive odd integer,

$$f(x, m) = 0$$

identically, because the terms equi-distant from the beginning and end cancel each other.

(*b*) If m is a positive even integer,

$$f(x, m) = (1 - x^{m-1})(1 - x^{m-3}) \ldots\ldots (1 - x) f(x, 0)$$

$$= (1 - x^{m-1})(1 - x^{m-3}) \ldots\ldots (1 - x).$$

(*c*) If m is a finite number and $|x| > 1$,

$$f(x, m) = (1 - x^{m-1})(1 - x^{m-3}) \ldots\ldots (1 - x^{m-2r+1}) f(x, m - 2r)$$

where

$$f(x, m - 2r) = 1 - \frac{1 - x^{m-2r}}{1 - x} + \frac{(1 - x^{m-2r})(1 - x^{m-2r-1})}{(1 - x)(1 - x^2)} - \ldots$$

$$= 1 + \frac{x^{-1}}{1 - x^{-1}} + \frac{x^{-3}}{(1 - x^{-1})(1 - x^{-2})}$$

$$+ \ldots + \frac{x^{-n(n+1)/2}}{(1 - x^{-1}) \ldots (1 - x^{-n})} + \ldots \text{ when } r \to \infty$$

$$= \prod_{n=1}^{\infty} (1 + x^{-n})$$

It is well-known that if

$$Q_1 = \Pi (1 + q^{2n}),$$

$$Q_2 = \Pi (1 + q^{2n-1}),$$

$$Q_3 = \Pi (1 - q^{2n-1})$$

where $|q| < 1$, then

$$Q_1 Q_2 Q_3 = 1.$$

Thus $$\frac{1}{\Pi (1 - q^{2n-1})} = \Pi (1 + q^n)$$

Hence it follows that

$$f(x, m) = \prod_{r=1}^{\infty} \left[\frac{1 - x^{m-2r+1}}{1 - x^{-2r+1}} \right].$$

(*d*) If $m = -\infty$ and $|x| > 1$,

$$f(x, m) = 1 + \sum_{n=1}^{\infty} \frac{x^{-n(n+1)/2}}{(1 - x^{-1})(1 - x^{-2}) \ldots\ldots (1 - x^{-n})}$$

$$= \prod_{n=1}^{\infty} (1 + x^{-n})$$

(e) If $|x| \to 1$ in case of each term,

$$f(x, m) = 1 - m + \frac{m(m-1)}{2!} - \frac{m(m-1)(m-2)}{3!} + \ldots\ldots$$

which converges absolutely or diverges according as $m \gtrless 0$.

(Case of the Binomial Theorem.)

(*f*) If $m = +\infty$ and $|x| < 1$,

$$f(x, m) = 1 - \frac{1}{1-x} + \frac{1}{(1-x)(1-x^2)}$$

$$- \frac{1}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$

in which the terms increase and $u_n \to l > 1$.

The series is non-convergent. (*Vide*: Bromwich, Ch. XI).

(*g*) If m is finite and $|x| < 1$,

$$f(x, m) = (1-x^{m-1})(1-x^{m-3})\ldots(1-x^{m-2r+1}).f(x, m-2r)$$

where $$f(x, m-2r) = 1 - \frac{1}{1-x} + \frac{1}{(1-x)(1-x^2)}$$

$$- \frac{1}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$

when $r \to \infty$ which is (*f*).

The other cases need not be considered.

[*Partial solutions by K. Satyanarayana and M. V. Ramakrishnan.*]

Question 1265.

(A. A. KRISHNASWAM IYENGAR) :—Show that every odd number can be expressed as the sum of seven squares (excepting 1, 3, 5, 9, 11, 17).

Remarks by 'Anon.'

In the solution published at page *147* of the June number (J.I.M.S., Vol. XV, No. 9) the following expression occurs :—" Let p be any prime contained in the form $(3^{2x+1}.n - 2)$." From the summary in Dickson's *History of the Theory of Numbers* (Vol. II, p. 290), it appears that Sylvester meant that the form $(3^{2x+1}.n - 2)$ includes "*composite as well as prime numbers.*" The abovementioned expression must therefore be understood in the sense that p is a prime *of the form* $(3^{2x+1}.n - 2)$, and *equal* to it; not merely a *factor* of it.

The enunciation of Q. 1265 is unnecessarily restricted. Since every positive integer can be expressed as the sum of *four* squares (some of which may be zero), it follows that the number

$$N - \sum_{1}^{k} (a_r)^2 \equiv \text{sum of } four \text{ squares.}$$

$$\therefore \qquad N \equiv \text{sum of } (k+4) \text{ squares,}$$

where the a's can be arbitrarily chosen subject to the obvious limitation that the sum of their squares is less than N. The theorem enunciated in the question is a very special case of the above.

Question 1266.

(A. A. KRISHNASAMY IYENGAR) :—Find general expressions for the sides of rational triangles the squares of whose areas are perfect cubes. (Ex.: 5, 6, 7.)

Solution by I. Totadri Iyengar.

The square of the area of a triangle ABC is

$$\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{16}.$$

To get the simplest expression, we put

$$a+b+c = 2p^2$$
$$a+b-c = 2q$$
$$b+c-a = 2q^2$$
$$c+a-b = 2p.$$

The relation satisfied by p and q is

$$p^2 = p + q + q^2.$$

or

$$p = q + 1.$$

$$\therefore \quad a+b+c = 2(q+1)^2$$
$$a+b-c = 2q$$
$$b+c-a = 2q^2$$
$$c+a-b = 2(q+1),$$

whence $a = 2q+1$, $b = q^2+q$, and $c = q^2+q+1$.

Question 1270.

(R. VAIDYANATHASWAMI) :—If a quadric has double three-point contact with a twisted cubic, show that it touches all the osculating planes of the curve.

Solution by Hemraj.

The parametric equations of a twisted cubic may be written

$$x : y : z : t = \theta^3 : \theta^2 : \theta : \tfrac{1}{3}.$$

The planes $x = 0$, $t = 0$ are the osculating planes at $\theta = 0$ and $\theta = \infty$ respectively.

The quadric $ax^2 + by^2 + cz^2 + dt^2 + 2fyz + 2gzx$

$$+ 2hxy + 2lxt + 2myt + 2nzt = 0$$

meets the cubic at $\theta = 0$ and $\theta = \infty$, each in three consecutive points

if $\quad a = 0,\ h = 0,\ d = 0,\ n = 0,\ b + 2g = 0,\ c + \frac{2m}{3} = 0;$

and the tangent planes to the quadric at $\theta = 0$, $\theta = \infty$ are the osculating planes thereat, if

$$a = 0 \quad h = 0 \quad g = 0 \quad m = 0 \quad n = 0 \quad d = 0.$$

$\therefore \qquad b = 0$ and $c = 0$. ... ... (1)

Hence the equation of the quadric having double three-point contact with the cubic at $\theta = 0$, $\theta = \infty$ is

$$2fyz + 2lxt = 0. \qquad \text{... ... (2)}$$

The tangent plane at any point $(x'y'z't')$ to the surface is

$$lt'x + fz'y + fy'z + lx't = 0. \qquad \text{... ... (3)}$$

The osculating plane at any point (θ) of the curve is

$$\frac{x}{3} - \theta y + \theta^2 z - \theta^3 t = 0. \qquad \text{... ... (4)}$$

(3) and (4) are identical if

$$3lt' = \frac{fz'}{-\theta} = \frac{fy'}{\theta^2} = \frac{lx'}{-\theta^3}$$

i.e., if $\quad 3l^2x't' = f^2y'z'$. But $fy'z' + lx't' = 0$.

Hence if $3l + f = 0$, then (2) touches all the osculating planes, as is otherwise obvious.

The tangential equation of the quadric is

$$AX^2 + BY^2 + CZ^2 + DT^2 + 2FYZ + 2GZX + 2HXY$$
$$+ 2LXT + 2MYT + 2NZT = 0.$$

The co-ordinates of any osculating plane are

$$X : Y : Z : T = \tfrac{1}{3} : -\theta : \theta^2 : -\theta^3.$$

It follows that osculating planes touch the quadric for all values of θ, if

$$A = 0,\ H = 0,\ D = 0,\ N = 0,\ B + \tfrac{2}{3}G = 0,$$

$$C + 2M = 0,\ F + \tfrac{1}{3}L = 0.$$

All these are satisfied except

$$F + \tfrac{1}{3}L = 0$$

when (1) holds.

$$F + \tfrac{1}{3}L = 0$$

is satisfied only if $l + \frac{1}{3}f = 0$.

Hence (2) takes the form $3yz - xt = 0$.

It follows that any quadric having merely double three-point contact with a twisted cubic, cannot be touched by all the osculating planes of the curve.

Now the quadric $3yz = xt$ satisfies all the conditions of the question, any twisted cubic can be transformed linearly into any other and all twisted cubics have the same projective properties. Hence the result.

Question 1271.

(K. SATYANARAYANA) :—(i) p being any odd prime, prove that

$$2^{p-1} - 1 \not\equiv 0 \pmod{p^2}.$$

Hence or otherwise establish that the numerator of

$$\left[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots \pm \frac{1}{\frac{1}{2}(p-1)}\right] \not\equiv 0 \pmod{p}.$$

Remarks by 'Anon.'

The question is incorrect. It has been known for a long time that the congruence

$$2^{p-1} \equiv 1 \pmod{p^2},\ p \text{ a prime}$$

has a solution $p = 1093$.

To verify this, we have

$$2^{14} = 16384 = -11 + 15\,p.$$

$$\therefore \quad 2^{28} \equiv 121 - 330\,p \ (\text{mod. } p^2)$$

and
$$\left.\begin{aligned} 9.2^{28} &\equiv 1089 - 2970\,p \\ &\equiv -4 - 1876\,p \end{aligned}\right\} ;$$

or
$$9.2^{26} \equiv -(1 + 469\,p).$$

Again,
$$\left.\begin{aligned} 9^7 . 2^{26\cdot7} &\equiv -(1 + 469\,p) \\ &\equiv -(1 + 3283\,p) \\ &\equiv -(1 + 4\,p) \end{aligned}\right\} ;$$

and
$$\left.\begin{aligned} 9^7 &\equiv (3^7)^2 \equiv (2187)^2 \\ &\equiv (1 + 2p)^2 \equiv 1 + 4\,p. \end{aligned}\right\}$$

Substituting, we get

$$2^{182} \equiv -1 \ (\text{mod. } p^2)$$

and
$$2^{1092} = 2^{182\cdot6} \equiv 1;$$

i.e.,
$$(2^{p-1} - 1) \equiv 0 \ (\text{mod } p^2). \text{ for } p = 1093.$$

N.B.—It has been verified (*vide*: any detailed account of Fermat's last Theorem) that this is the only solution less than 2000. No solution greater than 2000 is known. Perhaps some of our readers will find one.

Question 1273.

(S. AUDINARAYANAN) :—Fill up the vacant cells in the following square with integers in G. P. such that the continued product of the numbers along each row, column or diagonal is the same.

1			
			2

Solution by Mrs. Edith Thomas.

The common ratio of the G. P. is evidently 2.

Since the product $2^a . 2^b . 2^c . 2^d$, $= 2^{a+b+c+d}$ the question reduces to the selection of sets of four numbers from among the group

$$2^0,\ 2^1,\ 2^2,\ 2^3,\ \ldots\ldots\ 2^{15},$$

such that the sum of the indices of the four numbers in any set may be the same for all the sets.

Arranging the indices 0, 1, 2, 3, ... 15 in the following manner :—

0,	9,	6,	15
11,	7,	8,	4
14,	2,	13,	1
5,	12,	3,	10

it is found that the sum along any row, column or diagonal is the same, namely 30.

Hence the corresponding numbers required to fill in the vacant cells of the given square are

1,	2^9,	2^6,	2^{15}
2^{11},	2^7,	2^8,	2^4
2^{14},	2^2,	2^{13},	2
2^5,	2^{12},	2^3,	2^{10}

Thus the continued product in any direction $= 2^{30}$.

Other arrangements of the indices suggested by N. P. Pandya, Hemraj and N. Maitra respectively are :—

0	12	7	11
15	3	8	4
10	6	13	1
5	9	2	14

0	11	7	12
14	5	9	2
13	6	10	1
3	8	4	15

0	3	13	14
8	11	5	6
15	4	10	1
7	12	2	9

Questions for Solution.

1346. (S. D. CHOWLA) :—If $|x|<1$, prove that

$$\left\{\frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)\ldots\ldots}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots\ldots}\right\}^8 = \sum_{r=1}^{\infty}\left(\frac{r^3x^{r-1}}{1-x^{2r}}\right).$$

Deduce that the number of representations of an odd number as the sum of eight squares is 16 times the sum of the cubes of its divisors.

1347. (S. N. KUMARASWAMY) :—ABC is any triangle and P a given point within it. Show how to inscribe a triangle in it having the point P for

(i) centroid, (ii) orthocentre, (iii) incentre, (iv) symmedian point, (v) circumcentre.

1348. (S. D. CHOWLA) :—Prove that

$$\text{(i)} \quad \frac{e^{-\pi}}{1+e^{-\pi}}+\frac{3e^{-3\pi}}{1+e^{-3\pi}}+\frac{5e^{-5\pi}}{1+e^{-5\pi}}+\ldots\ldots=\frac{1}{24};$$

$$\text{(ii)} \quad \frac{e^{-2\pi}}{1-e^{-2\pi}}+\frac{2^5e^{-4\pi}}{1-e^{-4\pi}}+\frac{3^5e^{-6\pi}}{1-e^{-6\pi}}+\ldots\ldots=\frac{1}{504}.$$

1349. (V. TIRUVENKATACHARYA) :—Show that in 'hyperbolic geometry' the volume of a cylinder of radius a and length x is given by

$$V = \pi k^2 x \sinh^2\left(\frac{a}{k}\right)$$

where k is the space constant.

1350. (V. A. MAHALINGAM) :—Examine the correctness of the following—

"If $u_1, u_2, u_3, \ldots$ are solutions of

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

then their product $(u_1 u_2 u_3 \ldots)$ is also a solution of the same."

(Whittaker and Watson: *Modern Analysis*).

1351. (S. N. KUMARASWAMY) :—Find three numbers such that their sum is a perfect cube and the sum of their cubes is a perfect fourth power.

1352. (K. J. SANJANA) :—Show how to find integers less than given integer N and having the greatest possible number of divisors.

Example: $N = 150{,}000$.

1353. (ENQUIRER) :—Show that

$$\int_0^\infty \frac{\sin 2nx \sin^m x}{x^{a+1}}\, dx = \frac{\pi\, \Delta^m (2n-m)^a}{2^{m+1}\, \Gamma(a+1) \cos \frac{a+m}{2} \pi},$$

if $2n > m > a$, all being positive, and Δ is the operator in Finite-differences, equivalent to

$(E-1)$ or $(e^D - 1)$. [Boole's *Finite Differences.*]

1354. (K. J. SANJANA) :—P and Q are points on an ellipse whose eccentric angles are θ and ϕ; P′ and Q′ are the corresponding points on the nth pedal of the ellipse with respect to the focus S. Prove that

$$\left(\frac{SP}{SP'}\right)^{\frac{2}{n}} + \left(\frac{SQ}{SQ'}\right)^{\frac{2}{n}} = \frac{2 - e^2(\cos^2\theta + \cos^2\phi)}{1-e^2}.$$

1355. (MARTYN M. THOMAS) :—Show that

$$\int \left[\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{xdy - ydx}{uv}\right] = 2 \log \frac{v}{u},$$

where $u \equiv ax^2 + 2bxy + by^2$, $v = Ax^2 + 2Hxy + By^2$.

1356. (R. VAIDYANATHASWAMY) :—Prove the following construction for finding the point of contact of the pedal line of a point P with its envelope: Let O be the orthocentre and PQ the chord parallel to the pedal line of P. Produce QP to Q′ so that PQ′ = PQ. The required point of contact is the intersection of the pedal line with OQ′.

Is this construction known ?

1357. (R. VAIDYANATHASWAMY) :—A, B are two points in a plane, so related that when A is fixed, the locus of B is the circle SAA′, where A′ is the point such that SA = SA′ and $A\hat{S}A'$ = a fixed angle θ, S being a fixed point. If B is fixed, what is the locus of A ?

1358. (V. RAMASWAMI AIYAR) :—Four mutually orthocentric points A, B, C, D are the centres of four mutually orthogonal circles. If any point P invert with respect to these circles into P′, Q′, R′, S′ then the set [P′, Q′, R′, S′] will invert with respect to each of these circles into the same set of points [P, Q, R, S]. If G, G′ be the centroids of the sets (P, Q, R, S) and (P′, Q′, R′, S′) show that G, G′ are inverses of one another with respect to the nine-point circle of the system (A, B, C, D).

Examine what happens to the theorem if P were on one of the circles, say, the circle with centre A.

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LIST OF JOURNALS RECEIVED

(From 3rd August to 25th September 1924).

1 Bulletin of the Calcutta Mathematical Society, Vol. 13, No. 4.

2 Proceedings of the London Mathematical Society, Series 2, Vol. 23, Parts 2, 3.

3 Nature, Vol. 114, Nos. 2852, 2853, 2854, 2855, 2856, 2857, 2858, 2859.

4 The American Mathematical Monthly, Vol. 31, No. 6.

5 The Messenger of Mathematics, Vol. 53, Nos. 10, 11 and 12, Vol. 54, No. 1.

6 The Mathematical Gazette, Vol. 12, No. 171.

7 Proceedings of the Royal Society, Vol. 106, Nos. A. 735, 736.

8 The Astrophysical Journal, Vol. 59, No. 5, Vol. 60, No. 1.

9 Monthly Notices of the Royal Astronomical Society, Vol. 84, Nos. 7, 8.

10 The Philosophical Magazine, Vol. 48, No. 284.

11 Bulletin of the American Mathematical Society, Vol. 30, No. 7.

12 Popular Astronomy, Vol. 32, No. 7.

13 Revista Mathematica Hispano-Americana, Tome V, No 6 Tome VI, Nos. 1 to 6.

14 Annals of Mathematics, Vol. 25, No. 2.

15 Quarterly Journal of Mathematics, Vol. 50, No. 2.

16 Transactions of the Cambridge Philosophical Society, Vol. 23, No. 3.

Also the following books have been received :—

G. H. Halphen :—Œuvres, Tome IV.

H. A. Bender :—Sylow Subgroups in the Group of Isomorphisms of Prime Power Abelain Groups. *Thesis.*

W. E. Edington :—Abstract Group Definitions and Applications. *Thesis.*

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